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Fixed Point Theorems for Weak (o)-Contractive Multivalued Mappings

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Abstract

In this paper are given fixed point theorems for a class of multivalued mappings called weak (o)-contractive in ordered vector spaces, respectively ordered topological vector spaces, which generalize known results.

Key words: *ordered vector space, multivalued mapping, fixed point*

Weak (o)-Contractive Multivalued Mappings

Let X be an ordered vector space with the notations and terminology from [1]. We revisit some of the notions.

A sequence $(x_n)_{n\geq 0}$ with $x_n \in X$ is called *(o)-convergent* to $x \in X$ if there exist two sequences: one sequence $(a_n)_{n\geq 0}$ increasing and one sequence $(b_n)_{n\geq 0}$ decreasing such that

$$a_n \leq x_n \leq b_n\,,\ \forall n \geq 0$$

and

$$\sup\left\{a_n \,\middle|\, n \geq 0\right\} = \inf\left\{b_n \,\middle|\, n \geq 0\right\} = x\,.$$

We write $(o) - \lim\limits_{n\to\infty} x_n = x$ or shorter $x_n \xrightarrow{o} x$.

The ordered vector space X is *Archimedian* if and only if $\inf\left\{\frac{1}{n}x \,\middle|\, n \geq 1\right\} = 0$ for any $x \in X$ with $x > 0$. An ordered vector space X for which the ordered set X is directed to the right is called *directed vector space*.

Let X be an ordered vector space and $P(X)$ the family of non-empty subsets of X.

We consider mappings of type $f : X \to P(X)$ called *multivalued mappings*. An element $w \in X$ with $w \in f(w)$ is called *fixed point* of the multivalued mapping f.

Definition 1([2])**.** A multivalued mapping $f: X \to P(X)$ is called *(o)-contraction* if there exists $k \in (0,1)$ such as for any x, y, z from X with $-z \le x - y \le z$ and for any $u \in f(x)$ there exists $v \in f(y)$ with the property

$$-kz \le u - v \le kz .$$

Fixed point theorems for such multivalued functions and also for some of their extensions are given in [2], [3], [4], and [5].

In the following sections we consider weaker (o)-contraction conditions and we give fixed point theorems in ordered vector spaces and also in ordered topological vector spaces.

Definition 2. A multivalued mapping $f: X \to P(X)$ will be called *weak (o)-contractive* if there exists $q \in (0,1)$ such as for any $x \in X$, any $y \in f(x)$ and any $a \in X$ with $-a \le x - y \le a$, there exists $z \in f(y)$ such that

$$-qa \le y - z \le qa .$$

Remark 1. Any multivalued (o)-contractive mapping $f: X \to P(X)$ is weak (o)-contractive but not reciprocally.

Proof. The first part of the affirmation is obvious, the second part is proved by the following example.

Let $f: \mathbf{R} \to P(\mathbf{R})$ defined like this

$$f(x) = \begin{cases} (0, x), x > 0 \\ \{1\}, x = 0 \\ (x, 0), x < 0 \end{cases} .$$

It is easy to check that f is weak (o)-contractive with $q = \frac{1}{2}$, but f is not (o)-contractive because for $x = \frac{1}{2}$ and $y = 0$ we have $f\left(\frac{1}{2}\right) = \left(0, \frac{1}{2}\right)$, $f(y) = f(0) = \{1\}$. So v can only be 1 for any $u \in \left(0, \frac{1}{2}\right)$ and for example for $u = \frac{1}{4}$ it results that $v - u = \frac{3}{4}$ while $x - y = \frac{1}{2}$, which shows that f is not (o)-contraction.

We introduce a notion in ordered vector spaces corresponding to the one of closed graphic from the topological spaces, also some weaker ones.

Let X be an ordered vector space, $f: X \to P(X)$ a multivalued mapping and the following conditions:

(G) for any sequences $(x_n)_{n\ge 0}$, $(y_n)_{n\ge 0}$ from X with $x_n \xrightarrow{o} x$, $y_n \xrightarrow{o} y$, $y_n \in f(x_n)$, it results $y \in f(x)$

(G_1) for any sequences $(x_n)_{n\ge 0}$, $(y_n)_{n\ge 0}$ from X with $x_n \xrightarrow{o} x$, $y_n \xrightarrow{o} x$, $y_n \in f(x_n)$, it results $x \in f(x)$

(G_2) for any sequence $(x_n)_{n\ge 0}$ from X with $x_n \xrightarrow{o} x$, $x_{n+1} \in f(x_n)$, it results $x \in f(x)$

Evidently $(G) \Rightarrow (G_1) \Rightarrow (G_2)$.

If f satisfies condition (G) we will say that f has an *(o)-closed graphic*. We will say that a non-empty subset A of the ordered vector space X is *(o)-closed* if the limit of any (o)-convergent sequence from A is from A. We denote by $P_{oc}(X)$ the family of all the non-empty and (o)-closed subsets of the ordered vector space X.

Proposition 1. Let X be an Archimedian directed vector space. Then any multivalued (o)-contractive mapping $f : X \to P_{oc}(X)$ has an (o)-closed graphic.

Proof. Let $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0}$ be sequences from X with $x_n \xrightarrow{o} x$, $y_n \xrightarrow{o} y$ and $y_n \in f(x_n)$.

X is directed so from $x_n \xrightarrow{o} x$ it results that there is an decreasing sequence $(a_n)_{n\geq 0}$ with $a_n \xrightarrow{o} 0$ and

$$-a_n \leq x_n - x \leq a_n,\ \forall n \geq 0. \tag{1}$$

For $n = 0$, we obtain

$$-a_0 \leq x_0 - x \leq a_0.$$

Since $x_0, x \in X$, $y_0 \in f(x_0)$ and f is (o)-contraction, there exists $w_0 \in f(x)$ such that

$$-ka_0 \leq y_0 - w_0 \leq ka_0.$$

Than, for $n = 1$, (1) becomes

$$-a_1 \leq x_1 - x \leq a_1$$

and since $x_1, x \in X$, $y_1 \in f(x_1)$ it results that there exists $w_1 \in f(x)$ such that

$$-ka_1 \leq y_1 - w_1 \leq ka_1.$$

Continuing this procedure we obtain a sequence $(w_n)_{n\geq 0}$ with $w_n \in f(x)$ and

$$-ka_n \leq y_n - w_n \leq ka_n,\ \forall n \geq 0,$$

from where we deduce

$$(o) - \lim_{n\to\infty} w_n = (o) - \lim_{n\to\infty} y_n = y.$$

So we have $w_n \xrightarrow{o} y$, $w_n \in f(x)$, $f(x)$ (o)-closed which gives us $y \in f(x)$.

Remark 2. Weak (o)-contractive multivalued mappings do not necessarily verify the (G_2) condition and even more the (G) condition, meaning they do not necessarily have (o)-closed graphic.

Indeed, the multivalued mapping from the earlier example is weak (o)-contractive but does not satisfy the (G_2) condition because for the sequence $(x_n)_{n\geq 1}$ with $x_n = \frac{1}{n}$ we have $x_n \xrightarrow{o} 0$, $x_{n+1} = \frac{1}{n+1} \in f(x_n) = f\left(\frac{1}{n}\right) = \left(0, \frac{1}{n}\right)$, $f(0) = \{1\}$ and since $0 \notin f(0)$ it results that f does not satisfy (G_2).

Fixed Points for Weak (o)-Contractions in Ordered Vector Spaces

A sequence $(x_n)_{n\geq 0}$ from an ordered vector space is called *(o)-Cauchy* if there are the sequences: $(a_n)_{n\geq 0}$ increasing, $(b_n)_{n\geq 0}$ decreasing such that

$$a_n \leq x_n - x_{n+p} \leq b_n,\ \forall n, p \in \mathbf{N}$$

and

$$\sup\{a_n \,|n \geq 0\} = \inf\{b_n \,|n \geq 0\} = 0 .$$

An ordered vector space in which any (o)-Cauchy sequence is (o)-convergent is called *(o)-complete*.

Theorem 1. Let X be an Archimedian (o)-complete directed vector space. Then any weak (o)-contractive multivalued mapping $f : X \to P(X)$, having an (o)-closed graphic, has at least a fixed point.

Proof. Let $x_0 \in X$ be arbitrary, fixed. If $x_0 \in f(x_0)$ then the theorem is proved. If $x_0 \notin f(x_0)$ then there exists $x_1 \in f(x_0)$. Since X is a directed space there exists $a \in X$ such that

$$-a \leq x_0 - x_1 \leq a .$$

Since the multivalued mapping f is weak (o)-contractive it results that there exists $x_2 \in f(x_1)$ such that

$$-qa \leq x_1 - x_2 \leq qa .$$

Then from $x_1 \in X$ and $x_2 \in f(x_1)$ it results that there exists $x_3 \in f(x_2)$ such that

$$-q^2 a \leq x_2 - x_3 \leq q^2 a$$

and continuing this procedure we obtain the sequence $(x_n)_{n\geq 0}$ with $x_{n+1} \in f(x_n)$,

$$-q^n a \leq x_n - x_{n+1} \leq q^n a ,\ \forall n \geq 0 ,$$

called the *sequence of successive approximations*, from where we deduce

$$-\frac{q^n}{1-q} a \leq x_n - x_{n+p} \leq \frac{q^n}{1-q} a ,\ \forall n, p \in \mathbf{N} . \tag{2}$$

Since X is Archimedian, from (2) it results that $(x_n)_{n\geq 0}$ is an (o)-Cauchy sequence, since the space is (o)-complete we deduce that $(x_n)_{n\geq 0}$ is (o)-convergent, i.e. there exists $w \in X$ with

$$(o) - \lim_{n\to\infty} x_n = w .$$

Since the graphic of the multivalued mapping f is supposed to be (o)-closed, from $x_n \to w$, $x_{n+1} \in f(x_n)$ it results that $w \in f(w)$, i.e. w is a fixed point of the multivalued mapping f .

The theorem is now completely proved.

Theorem 1 from [2] results as a consequence of theorem 1, previously proved.

Corollary 1. Let X be an Archimedean (o)-complete directed vector space. If $f : X \to P_{oc}(X)$ is a multivalued (o)-contractive mapping then f has a fixed point.

Indeed, counting on remark 1 f is weak (o)-contractive, and with proposition 1 f has a (o)-closed graphic.

Since in the proof of theorem 1 the property of (o)-closed graphic is used in the weak form (G_2), the following result is obtained:

Theorem 2. Let X be an Archimedean (o)-complete directed vector space. Then any weak (o)-contractive multivalued mapping $f : X \to P(X)$, which verifies (G_2), has a fixed point.

A consequence of this theorem is theorem 1 from [3].

Corollary 2. Let X be an Archimedean (o)-complete directed vector space. If the multivalued mapping $f : X \to P_{oc}(X)$ has the property that $\exists \alpha, \beta, \gamma, \rho$ non-negative real numbers with $\alpha + \rho = \beta + \gamma$, $1 + \gamma < \beta$ such that for any $x, y, a \in X$ with $-a \le x - y \le a$ and for any $u \in f(x)$ $\exists v \in f(y)$ which verifies the condition

$$-a \le \alpha u - \beta v - \gamma x + \rho y \le a,$$

then f has a fixed point.

Proof.

(a) f is weak (o)-contractive. Indeed if $u = y$ and we note $v = z$ it results

$$-a \le \alpha y - \beta z - \gamma x + \rho y \le a$$

or

$$-a \le (\beta + \gamma - \rho) y - \beta z - \gamma x + \rho y \le a$$

i.e.

$$-a \le \beta(y - z) - \gamma(x - y) \le a$$

so

$$\gamma(x - y) - a \le \beta(y - z) \le \gamma(x - y) + a$$

and counting on $-a \le x - y \le a$, we obtain

$$-\frac{1+\gamma}{\beta} a \le y - z \le \frac{1+\gamma}{\beta} a$$

where $0 < \frac{1+\gamma}{\beta} < 1$. Therefore f is weak (o)-contractive.

(b) f verifies (G_1).

Indeed, let $x_n \xrightarrow{o} x$, $y_n \xrightarrow{o} x$ with $y_n \in f(x_n)$. Since $x_n \xrightarrow{o} x$ it results that there exists a decreasing sequence $(a_n)_{n \ge 0}$ with $a_n \xrightarrow{o} 0$ and

$$-a_n \le x_n - x \le a_n, \ \forall n \ge 0.$$

Because $x_n, x \in X$, $y_n \in f(x_n)$ it results that there exists $w_n \in f(x)$ such that

$$-a_n \le \alpha y_n - \beta w_n - \gamma x_n + \rho x \le a_n, \ \forall n \ge 0,$$

from where we deduce

$$(o)-\lim_{n\to\infty}(\alpha y_n - \beta w_n - \gamma x_n + \rho x) = 0$$

or

$$\beta\left((o)-\lim_{n\to\infty} w_n\right) = (\alpha+\rho-\gamma)x$$

and since $\alpha+\rho-\gamma=\beta$ it results

$$(o)-\lim_{n\to\infty} w_n = x\,.$$

From $w_n \xrightarrow{o} x$, $w_n \in f(x)$, $f(x)$ is (o)-closed it results $x \in f(x)$.

Fixed Points for Weak (o)-Contractions in Ordered Topological Vector Spaces

A subset A of an ordered vector space is called *full* if from $x_1, x_2 \in A$ and $x_1 \le x_2$ it results $[x_1, x_2] \subset A$.

An ordered vector space X endowed with a vector topology τ with the property that there exists a neighborhood basis of the origin formed of full sets, is called *ordered topological vector space* and if additionally X is directed then X will be called *directed topological vector space*.

Theorem 3. Let X be a directed topological vector space, sequentially (τ)-complete with the cone $X_+ = \{x \in X \mid x \ge 0\}$ (τ)-closed. Then any weak (o)-contractive multivalued mapping $f : X \to P(X)$ with (τ)-closed graphic has at least a fixed point.

Proof. Let $x_0 \in X$ and $(x_n)_{n\ge 0}$ with $x_n \in f(x_{n-1})$ the corresponding sequence of the successive approximations build like in theorem 1. Since $\frac{q^n}{1-q} a \xrightarrow{\tau} 0$, from (2) it results that for any balanced and full neighborhood V of the origin in X, there exists $n_V \in \mathbf{N}$ such that

$$x_n - x_{n+p} \in V\,,\ \forall n \ge n_V \text{ and } \forall p \in \mathbf{N}\,,$$

that shows that $(x_n)_{n\ge 0}$ is a (τ)-Cauchy sequence and so (τ)-convergent, i.e. there exists $w \in X$ with $(\tau)-\lim_{n\to\infty} x_n = w$.

We have

$$x_n \xrightarrow{\tau} w\,,\ x_{n+1} \in f(x_n)$$

and since f has a (τ)-closed graphic it results that $w \in f(w)$.

Common Fixed Points for Pairs of Weak (o)-Contractive Multivalued Mappings

Definition 3 ([3]). Let X be an ordered vector space. It is said that the multivalued mappings $f,g : X \to P(X)$ forms a *pair of (o)-contractions* if there exists $k \in (0,1)$ such that for any $x, y, a \in X$ with $-a \le x - y \le a$ and for any $u \in f(x)$ $\left(u \in g(y)\right)$ there exists $v \in g(y)$ $\left(v \in f(x)\right)$ with $v \ne u$ which verifies the condition

$$-ka \le u - v \le ka .$$

Definition 4. We will say that the multivalued mappings $f,g : X \to P(X)$ forms a *weak (o)-contractive pair* if there exists $q \in (0,1)$ such that for any $x \in X$, any $y \in f(x)$ $\left(y \in g(x)\right)$, any $a \in X$ with $-a \le x - y \le a$ there exists $z \in g(y)$ $\left(z \in f(y)\right)$ with $z \ne y$ and which verifies the condition

$$-qa \le y - z \le qa .$$

Evidently any pair of (o)-contractions is a weak (o)-contractive pair.

Proposition 2. Let X be an Archimedian directed vector space and $f,g : X \to P_{oc}(X)$ a pair of (o)-contractions. Then f and g have an (o)-closed graphic.

Proof. Let $(x_n)_{n\ge 0}$, $(y_n)_{n\ge 0}$ be sequences from X such that $x_n \xrightarrow{o} x$, $y_n \xrightarrow{o} y$, $y_n \in f(x_n) \bigcap g(x_n)$. There exists $(a_n)_{n\ge 0}$ decreasing with $a_n \xrightarrow{o} 0$ such that

$$-a_n \le x_n - x \le a_n ,\ \forall n \ge 0 , \tag{3}$$

which for $n = 0$ is

$$-a_0 \le x_0 - x \le a_0 .$$

Since $x_0, x \in X$, $y_0 \in g(x_0)$ it results that there exists $w_0 \in f(x)$ such that

$$-ka_0 \le y_0 - w_0 \le ka_0 .$$

Then, for $n = 1$, (3) becomes

$$-a_1 \le x_1 - x \le a_1$$

and since $x_1, x \in X$, $y_1 \in f(x_1)$ it results that there exists $w_1 \in g(x)$ with

$$-ka_1 \le y_1 - w_1 \le ka_1 .$$

Continuing this procedure we obtain a sequence $(w_n)_{n\ge 0}$ with $w_{2n} \in f(x)$, $w_{2n+1} \in g(x)$ and

$$-ka_n \le y_n - w_n \le ka_n ,\ \forall n \ge 0 ,$$

from where it results

$$(o) - \lim_{n\to\infty} w_n = (o) - \lim_{n\to\infty} y_n = y$$

and since $f(x)$ and $g(x)$ are (o)-closed, we deduce $w \in f(x) \bigcap g(x)$.

Theorem 4. Let X be an Archimedian (o)-complete directed vector space. Then any weak (o)-contractive pair of multivalued mappings $f, g : X \to P(X)$ with (o)-closed graphics have at least a common fixed point, i.e. there exists $w \in X$ with $w \in f(x) \bigcap g(x)$.

Theorem 1 from [3] results as a consequence of theorem 4.

Corollary 3. If X is an Archimedian (o)-complete directed vector space and $f, g : X \to P_{oc}(X)$ forms a pair of (o)-contractions, then they have at least a common fixed point.

Indeed, f , g forms a weak (o)-contractive pair and with proposition 2 they have (o)-closed graphics.

Theorem 5. Let X be a directed topological vector space, sequentially (τ)-complete with the (τ)-closed cone X_+. Then any weak (o)-contractive pair of multivalued mappings $f, g : X \to P(X)$ with (τ)-closed graphics have at least a common fixed point.

Proof. The sequence of the successive approximations $(x_n)_{n\geq 0}$ is build like in the proof of theorem 4 with $x_{2n+1} \in f(x_{2n})$, $x_{2n+2} \in g(x_{2n+1})$. From (4) we obtain $(\tau) - \lim_{n\to\infty} x_n = w$ and than we continue with thinking alike theorem 3's proof, which leads to $w \in f(x) \bigcap g(x)$.

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Teoreme de punct fix pentru multifuncţii slab (o)-contractive

Rezumat

În această lucrare se dau teoreme de punct fix pentru o clasă de multifuncţii numite slab (o)-contractive în spaţii liniare ordonate, respectiv spaţii liniare ordonate topologice, care generalizează rezultate cunoscute.